APPROXIMATE ACOUSTIC CLOAKING IN INHOMOGENEOUS ISOTROPIC SPACE

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Abstract. In this paper, we consider the approximate acoustic cloaking in inhomogeneous isotropic background space. By employing transformation media, together with the use of a sound-soft layer lining right outside the cloaked region, we show that one can achieve the near-invisibility by the 'blow-up-a-small-region' construction. This is based on novel scattering estimates corresponding to small sound-soft obstacles located in isotropic space. One of the major novelties of our scattering estimates is that one cannot make use of the scaling argument in the setting of current study due to the simultaneous presence of asymptotically small obstacle components and regularly sized obstacle components, and one has to decouple the nonlinear scattering interaction between the small obstacle components and, the regular obstacle components together with the background medium.

1. Introduction

A region is said to be *cloaked* if its contents together with the cloak are indistinguishable from the background space to certain exterior detections. Blueprints for making objects invisible to electromagnetic waves were proposed by Pendry *et al.* [22] and Leonhardt [16] in 2006. In the case of electrostatics, the same idea was discussed by Greenleaf *et al.* [10] in 2003. The key ingredient for those constructions is that optical parameters have transformation properties and could be *push-forwarded* to form new material parameters. The obtained materials/media are called *transformation media*, which we shall further examine in the current work for approximate acoustic cloaking in inhomogeneous isotropic space.

The transformation media proposed in [10, 22] are rather singular. This poses much challenge to both theoretical analysis and practical fabrication. In order to avoid the singular structures, several regularized approximate cloaking schemes are proposed in [7, 13, 14, 17, 23]. The basic idea is to introduce regularization into the singular transformation underlying the ideal cloaking, and instead of the perfect

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invisibility, one would consider the 'near-invisibility' depending on the regularizer. The works [7] and [23] are based on truncation, whereas in [13, 14, 17], the 'blow-up-a-point' transformation in [10, 22] is regularized to be the 'blow-up-a-small-region' transformation. The performances of both regularization schemes have been assessed for cloaking of acoustic waves to give successful near-invisibility effects. Particularly, in [13], the authors show that in order to 'nearly-cloak' an *arbitrary* content, it is necessary to employ an absorbing ('lossy') layer lining right outside the cloaked region. Since otherwise, there exist cloaking-busting inclusions which defy any attempts of cloaking. If one lets the lossy parameter go to infinity, heuristically we would have the sound-soft material lining, which is the one considered in [17]. All the aforementioned studies for approximate acoustic cloaking are conducted in the homogeneous background space, and there is no result available in literature for the more general case when the background space is allowed to be inhomogeneous. Nonetheless, it is worthy noting that the result in [14] for approximate cloaking of conductivity equation could be readily extended to the case with inhomogeneous background space by making use of the estimates in [6] for small extreme conductivities; whereas the result in [13] could also be readily extended to the case with inhomogeneous background space by making use of the low-frequency estimates in [3] for the reduced wave equation. One of the key points for those extensions is that the corresponding studies could be reduced to the scattering estimates due to *uniformly* small objects, and then by scaling arguments, the studies could be further reduced to those having been considered in [6] and [3].

In this work, we shall consider the approximate cloaking for acoustic waves in a very general and practical setting when the background space is allowed to be inhomogeneous but isotropic. We are mainly interested in the practical case that in the inhomogeneous space, there are both target objects one intends to cloak and non-target objects being uncloaked. By transformation-optics-approach, we construct the approximate cloaking devices by the 'blow-up-a-small-region' scheme. In order to overcome the cloaking-busts due to resonance, we implement the sound-soft lining right outside of the cloaked region. In assessing the cloaking performance, the study is shown to be reduced to the scattering estimate due to extended objects with both asymptotically small obstacle components and regularly sized obstacle components being presented simultaneously in an inhomogeneous space. One cannot make use of the scaling arguments as mentioned earlier and has to decouple the nonlinear scattering interactions between the small obstacle components, the regular obstacle components and the

background medium. By nonlinear scattering interaction we mean that the interaction between the scattered wave fields due to different components in a scattering system is a nonlinear process. Similar case with the background space being homogeneous has been investigated in [17], where boundary integral equations method is used to decouple the scattering due to the small obstacle components and regular obstacle components. For the current study, the presence of the inhomogeneous medium makes the corresponding arguments rather technical. We derive a novel system of integral equations underlying the scattering problem, which combines the volume potential operator of Lippman-Schwinger type and single- and double- boundary layer potential operators. An asymptotical coupling parameter is also a crucial incorporation. Since the scaling arguments does not apply here, another major difficulty one need to handle is that the domain of the underlying PDE is always in change in the asymptotic analysis. By extensive use of the potential operators theory, we show that the scattering contribution from small obstacle components is also asymptotically small with respect to their sizes, which justifies the near-cloak.

In this paper, we focus entirely on transformation-optics-approach in constructing cloaking devices. We refer to [8, 9, 21, 25, 26] for state-of-the-art surveys on the rapidly growing literature and many striking applications of transformation optics. But we would also like to mention in passing the other promising cloaking schemes including the one based on anomalous localized resonance [20], and another one based on special (object-dependent) coatings [1].

The rest of the paper is organized as follows. In Section 2, we give a brief discussion on inverse acoustic scattering and invisibility cloaking. In Section 3, we collect the basics on transformation optics and apply them to the construction of approximate cloaking devices. Sections 4 is devoted to the scattering estimates due to extended objects and the proof of the main result on approximate cloaking.

2. Acoustic scattering and invisibility cloaking

We first fix some notations that shall be used throughout the rest of the paper. For two domains D and Ω in $\mathbb{R}^n$ (n=2,3), $D \Subset \Omega$ means that $D \subset \bar{D} \subset \Omega$. χ_D denotes the characteristic function of D. Also, B_r is reserved for a central ball of radius r in $\mathbb{R}^n$. For two relations $\mathcal{R}_1$ and $\mathcal{R}_2$, $\mathcal{R}_1 \lesssim \mathcal{R}_2$ will refer to $\mathcal{R}_1 \leq c\mathcal{R}_2$ with c a generic constant which may change in different inequalities but must be fixed and finite in a given relation. $\mathcal{R}_1 \sim \mathcal{R}_2$ means that we have both $\mathcal{R}_1 \lesssim \mathcal{R}_2$ and $\mathcal{R}_2 \lesssim \mathcal{R}_1$. We shall also let $c(i_1, i_2, \ldots, i_l)$ denote a generic constant

that depends on the ingredients $i_1, i_2, \ldots, i_l$. The notations of function spaces is the usual one, e.g., $L^2(\Omega)$ denotes the space of square integrable functions on Ω, and $H^s(\Omega)$ denotes the Sobolev space of order s on Ω.

Let $\sigma = (\sigma^{ij})_{i,j=1}^n \in \mathrm{Sym}_n$ be a symmetric-matrix-valued function on $\mathbb{R}^n$ and bounded in the sense that, for some constants $0 < c_0, C_0 < \infty$,

$$c_0 \xi^T \xi \leq \xi^T \sigma(x) \xi \leq C_0 \xi^T \xi \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. Also, we let $\eta \in L^\infty(\mathbb{R}^n)$ be a scalar function with

$$\eta(x) \geq \eta_0 > 0 \quad \forall\, x \in \mathbb{R}^n. \tag{2.2}$$

In acoustic scattering, σ^{-1} and η, respectively, represent the mass density tensor and the bulk modulus of a *regular* acoustic medium. Starting from now on, we denote by $\{\mathbb{R}^n; \sigma, \eta\}$ an acoustic medium as described above. We shall assume that the inhomogeneity of the acoustic medium is compactly supported, namely, $\sigma = I$ and $\eta = 1$ in $\mathbb{R}^n \backslash \bar{\mathbf{B}}$ with $\mathbf{B}$ a bounded domain in $\mathbb{R}^n$. In $\mathbb{R}^n$, the time-harmonic acoustic wave propagation is governed by the heterogeneous Helmholtz equation

$$\sum_{i,j=1}^n \partial_i(\sigma^{ij}\partial_j u) + \omega^2 \eta u = f, \tag{2.3}$$

where *supp* $f \Subset \mathbf{B}$ represents a compactly supported source/sink. Here, $\omega > 0$ represents the frequency of the wave scattering. In the following, we shall write $\{\mathbf{B}; \sigma, \eta, f\}$ to denote the scattering object including the inhomogeneous medium and the source/sink supported in Ω. We seek solutions admitting the following asymptotic development as $|x| \to +\infty$

$$u(x) = e^{ix\cdot\xi} + \frac{e^{i\omega|x|}}{|x|^{(n-1)/2}} \left\{ A(\theta, \theta', \omega) + \mathcal{O}(\frac{1}{|x|}) \right\}, \tag{2.4}$$

where $\theta, \theta' \in \mathbb{S}^{n-1}$ and $\xi = \omega\theta'$. $A(\theta, \theta', \omega)$ is known as the *scattering amplitude*, which depends on the direction θ' and frequency ω of the incident wave $u^i := e^{ix\cdot\xi}$, observation direction θ, and obviously, also the underlying scattering object $\{\Omega; \sigma, \eta, f\}$. In inverse scattering theory, one intends to recover the target object $\{\mathbf{B}; \sigma, \eta, f\}$ by knowledge of $A(\theta, \theta', \omega)$ (cf. [4, 15]). In the sequel, we use $A(\{\mathbf{B}; \sigma, \eta, f\})$ to denote the scattering amplitude corresponding to $\{\mathbf{B}; \sigma, \eta, f\}$. In this context, an invisibility cloaking device is introduced as follows.

Definition 2.1. For a given regular background/reference medium $\{\mathbf{B}; \sigma_0, \eta_0\}$, let Ω and D be bounded domains such that $D \Subset \Omega \Subset \mathbf{B}$. $\Omega \backslash \bar{D}$ and D represent, respectively, the cloaking region and the cloaked

region. $\{\Omega\backslash\bar{D};\sigma_c,\eta_c\}$ is said to be an *invisibility cloaking device* for the region D with respect to the background space $\{\mathbf{B};\sigma_0,\eta_0\}$ if

$$A(\{\mathbf{B};\sigma_e,\eta_e,f_e\})=A(\{\mathbf{B};\sigma_0,\eta_0\}),$$

where the extended object $\{\mathbf{B};\sigma_e,\eta_e,f_e\}=\{D;\sigma_a,\eta_a,f_a\}\oplus\{\Omega\backslash\bar{D};\sigma_c,\eta_c\}$ $\oplus\{\mathbf{B}\backslash\bar{\Omega};\sigma_0,\eta_0\}$ with $\{D;\sigma_a,\eta_a,f_a\}$ arbitrary but regular. Here, we use $\oplus$ to concatenate separate scattering objects.

According to Definition 2.1, the cloaking medium $\{\Omega\backslash\bar{D};\sigma_c,\eta_c\}$ makes the target object $\{D;\sigma_a,\eta_a,f_a\}$ indistinguishable from the background space $\{\mathbf{B};\sigma_0,\eta_0\}$, and thus invisible to the scattering detections.

As can be seen from our subsequent discussion in Section 3, one has to implement singular cloaking medium in order to achieve the ideal cloaking (see also the related discussion in [13] and [7]). This poses many challenges to both mathematical analysis and physical realization. In order to construct practical nonsingular cloaking devices, it is natural to incorporate regularization by considering approximate cloaking. We conclude this section by introducing the notion of approximate acoustic cloaking.

Definition 2.2. Let $\{\mathbf{B};\sigma_0,\eta_0\}$, Ω and D be given as in Definition 2.1. Let $\rho>0$ denote a regularizer and $e(\rho)$ be a positive function such that

$$e(\rho)\to 0\quad\text{as }\rho\to 0^+.$$

$\{\Omega\backslash\bar{D};\sigma_c^\rho,\eta_c^\rho\}$ is said to be an *approximate invisibility cloaking* for the region D if

$$\|A(\{\mathbf{B};\sigma_e,\eta_e,f_e\})-A(\{\mathbf{B};\sigma_0,\eta_0\})\|=e(\rho)\quad\text{as }\rho\to 0^+,\tag{2.5}$$

where the extended object $\{\mathbf{B};\sigma_e,\eta_e,f_e\}$ is defined similarly to the one in Definition 2.1 by replacing $\{\Omega\backslash\bar{D};\sigma_c,\eta_c\}$ with $\{\Omega\backslash\bar{D};\sigma_c^\rho,\eta_c^\rho\}$.

According to Definition 2.2, with the cloaking device $\{\Omega\backslash\bar{D};\sigma_c^\rho,\eta_c^\rho\}$ we shall have the 'near-invisibility' cloaking effect within $e(\rho)$ depending on the regularizer ρ.

3. Transformation optics and cloaking by sound-soft layer lining

The transformation-optics-approach for the construction of cloaking devices critically relies on the following transformation invariance of the Helmh- oltz equation. Let $\tilde{x}=F(x):\Omega\to\tilde{\Omega}$ be a bi-Lipschitz and orientation-preserving mapping. For an acoustic medium $\{\Omega;\sigma,\eta\}$, we let the *push-forwarded* medium be

$$\{\tilde{\Omega};\tilde{\sigma},\tilde{\eta}\}=F_*\{\Omega;\sigma,\eta\}:=\{\Omega;F_*\sigma,F_*\eta\},\tag{3.1}$$

where

$$\begin{aligned}\tilde{\sigma}(\tilde{x}) &= F_*\sigma(x) := \frac{1}{J}M\sigma(x)M^T|_{x=F^{-1}(\tilde{x})}\\ \tilde{\eta}(\tilde{x}) &= F_*\eta(x) := \eta(x)/J|_{x=F^{-1}(\tilde{x})}\end{aligned} \tag{3.2}$$

and $M = (\partial\tilde{x}_i/\partial x_j)_{i,j=1}^n$, $J = \det(M)$. Assume that $u \in H^1(\Omega)$ is a solution to the Helmholtz equation associated with $\{\Omega; \sigma, \eta\}$, namely,

$$\nabla\cdot(\sigma\nabla u) + \omega^2\eta u = 0 \quad \text{on } \Omega,$$

then the pull-back field $\tilde{u} = (F^{-1})^* u := u \circ F^{-1} \in H^1(\tilde{\Omega})$ verifies

$$\tilde{\nabla}\cdot(\tilde{\sigma}\tilde{\nabla}\tilde{u}) + \omega^2\tilde{\eta}\tilde{u} = 0 \quad \text{on } \tilde{\Omega},$$

where we use ∇ and $\tilde{\nabla}$ to distinguish the differentiations respectively in x- and $\tilde{x}$-coordinates. We refer to [13, 17] for a proof of this invariance.

Next, we shall give the construction of the cloaking device. We first fix the setting for our study. Let $\{\mathbb{R}^n; \sigma_0, \eta_0\}$ be a regular background/reference space. Throughout, we shall assume that the background space medium is isotropic, i.e., σ_0 is a multiple of a scalar function and the identity matrix. In the following, we treat σ_0 as a scalar function which should be clear in the context. Let $\sigma_0(x) \in C^3(\mathbb{R}^n)$ and $\eta_0 \in C^1(\mathbb{R}^n)$, and $supp\ (1-\sigma_0), supp\ (1-\eta_0) \Subset \mathbf{B}$. Let $D, G \Subset \mathbf{B}$ be C^2-domains in $\mathbb{R}^n$ such that $\mathbb{R}^n\backslash(\bar{D}\cup\bar{G})$ is connected and D is of the form $D = \cup_{k=1}^l D_k$, where each D_k is simply connected. It is assumed that D_k's are disjoint and separate from G. Let $z_k \in D_k$ be points such that

$$|z_k - z_{k'}| \geq d_0 > 0, \quad \forall\ k \neq k', \tag{3.3}$$

and let P_k and O_k be C^2-domains such that P_k's and O_k's are simply connected and

$$z_k \in P_k \Subset D_k, \quad \mathbf{0} \in O_k.$$

Define

$$U = \bigcup_{k=1}^l U_k, \quad U_k = z_k + O_k. \tag{3.4}$$

Here, we assume that $\mathbb{R}^n\backslash(\bar{U}\cup\bar{G})$ is connected and

$$\mathrm{dist}_{\mathbb{R}^n}(\bar{U},\bar{G}) \geq r_0 > 0, \quad \mathrm{dist}_{\mathbb{R}^n}(\bar{U}_k,\bar{U}_{k'}) \geq \tilde{r}_0 > 0, \quad \forall\ k \neq k'. \tag{3.5}$$

For a sufficiently small $\rho > 0$, we let

$$U_\rho = \bigcup_{k=1}^l U_{k,\rho}, \quad U_{k,\rho} = z_k + \rho O_k. \tag{3.6}$$

Clearly, the parameter ρ determines the relative size of $U_{k,\rho}$. Let ρ be sufficiently small such that $U_{k,\rho} \Subset P_k$. We suppose that there

are orientation-preserving C^2-diffeomorphisms $F_k^\rho : \bar{D}_k\backslash U_{k,\rho} \to \bar{D}_k\backslash P_k$ such that

$$\bar{D}_k\backslash P_k = F_k^\rho(\bar{D}_k\backslash U_{k,\rho}), \quad F_k^\rho|_{\partial D_k} = \text{Identity}, \quad k = 1, 2, \ldots, l. \tag{3.7}$$

One sees that F_k^ρ blows up the small region $U_{k,\rho}$ to P_k within $\bar{D}_k$. Here, we would like to mention that one of such mappings is given by

$$\mathcal{F}(x) = (a + b|x|)\frac{x}{|x|}, \quad a = \frac{r_1 - \rho}{r_2 - \rho} r_2, \quad b = \frac{r_2 - r_1}{r_2 - \rho}, \quad r_2 > r_1 > 0,$$

which blows up the central ball of radius ρ to the central ball of radius r_1 within the central ball of radius r_2.

We are ready to construct the approximate cloaking device. Let P_k, $D_k\backslash\bar{P}_k$ denote respectively the cloaked and cloaking regions. The cloaking media are given by

$$\{D_k\backslash\bar{P}_k; \sigma_{k,c}, \eta_{k,c}\} = (F_k^\rho)^*\{D_k\backslash\bar{U}_{k,\rho}; \sigma_0, \eta_0\}, \quad 1 \le k \le l. \tag{3.8}$$

That is, we blow up a small region $U_{k,\rho}$ within D_k, and derive the cloaking medium in $D_k\backslash\bar{P}_k$ by pushing-forward the background medium in $D_k\backslash\bar{U}_{k,\rho}$. Next, one can put the target objects in P_k, namely, the passive *regular* media or radiating sources. The last ingredient for our cloaking construction is to introduce a thin sound-soft layer lining right outside P_k. Then, P_k can be regarded as a *sound-soft obstacle*. Here, by a sound-soft obstacle, we mean a scattering object which prevents acoustic wave penetrating inside (similarly, the wave propagation inside P_k cannot penetrate outside and completely trapped inside), and the wave pressure vanishes on its boundary; that is, one would the following homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \ \ \partial P_k.$$

For such construction, the target object together with its cloaking is

$$\mathcal{S} := \bigoplus_{k=1}^{l}\{D_k\backslash\bar{P}_k; \sigma_{k,c}, \eta_{k,c}\} \oplus P, \tag{3.9}$$

where $P = \cup_{k=1}^{l} P_k$ is the union of all target sound-soft obstacle components; and the extended scattering object is given by

$$\mathcal{E} = \mathcal{S} \oplus G \oplus \{\mathbf{B}\backslash(\bar{D} \cup \bar{G}); \sigma_0, \eta_0\} \tag{3.10}$$

The near-invisibility of our construction is given in the following theorem.

Theorem 3.1. *For any fixed $\xi \in \mathbb{R}^n$, we have as $\rho \to 0^+$,*

$$\|A(\theta, \xi; \mathcal{E}) - A(\theta, \xi; G \oplus \{\mathbf{B}\backslash\bar{G}; \sigma_0, \eta_0\})\|_{L^2(\mathbb{S}^{n-1})} = \mathcal{O}(e(\rho)), \tag{3.11}$$

where

$$e(\rho) = \begin{cases} \rho, & n = 3 \\ (\ln \rho)^{-1}, & n = 2 \end{cases} \tag{3.12}$$

Theorem 3.1 indicates that the cloaked components P shall be nearly cloaked, whereas the uncloaked component G remains unaffected, even though there is scattering interactions among different components.

We shall prove Theorem 3.1 in the next section, which is a consequence of the following theorem.

Theorem 3.2. *Let U_ρ and G be sound-soft obstacles and*

$$\mathcal{T} := U_\rho \oplus G \oplus \{\mathbf{B} \backslash (\bar{U}_\rho \cup \bar{G}); \sigma_0, \eta_0\}. \tag{3.13}$$

For any fixed $\xi \in \mathbb{R}^n$, we have as $\rho \to 0^+$

$$\|A(\theta, \xi; \mathcal{T}) - A(\theta, \xi; G \oplus \{\mathbf{B} \backslash \bar{G}; \sigma_0, \eta_0\})\|_{L^2(\mathbb{S}^{n-1})} = \mathcal{O}(e(\rho)), \tag{3.14}$$

where $e(\rho)$ is given in (3.12).

Theorem 3.2 indicates that for scattering due to obstacles in the space $\{\mathbf{B}; \sigma_0, \eta_0\}$, the scattering contribution from small obstacle components is also small in terms of their sizes. The proof of Theorem 3.2 will also be given in the next section. To our best knowledge, the result in Theorem 3.2 are completely new in literature. We would like to remark that the scattering estimates due to small inclusions have received extensive mathematical interests in literature (see, e.g. [2]). So, our result in Theorem 3.2 together with the techniques developed for its proof is of mathematical importance and significance for its own sake.

4. Scattering estimates of small obstacles in isotropic space

We first prove Theorem 3.2. The scattering problem corresponding to $\mathcal{T}$ in (3.13) is given by

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla u) + \omega^2 \eta_0 u = 0 \quad \text{in } \mathbb{R}^n \backslash (\bar{U}_\rho \cup \bar{G}), \\ u|_{\partial U_\rho \cup \partial G} = 0, \end{cases} \tag{4.1}$$

whereas the one without the small inclusions U_ρ is

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla \tilde{u}) + \omega^2 \eta_0 \tilde{u} = 0 \quad \text{in } \mathbb{R}^n \backslash \bar{G}, \\ \tilde{u}|_{\partial G} = 0, \end{cases} \tag{4.2}$$

Let $u^s(x) := u(x) - e^{ix\cdot\xi}$ and $\tilde{u}^s(x) := \tilde{u}(x) - e^{ix\cdot\xi}$ be the so-called *scattered wave fields*, and $\mathbf{w}^s$ denote either u^s or $\tilde{u}^s$. Then, it is well known that $\mathbf{w}^s$ satisfies the Sommerfeld radiation condition (cf. [4])

$$\lim_{|x|\to\infty} |x|^{(n-1)/2}\{\frac{\partial \mathbf{w}^s}{\partial |x|} - i\omega\mathbf{w}^s\} = 0. \tag{4.3}$$

We refer to [11, 12] for related study on the unique existence of $u \in H^1_{loc}(\mathbb{R}^n\backslash(\bar{U}_\rho \cup \bar{G}))$ to (4.1)-(4.3), and $\tilde{u} \in H^1_{loc}(\mathbb{R}^n\backslash\bar{G})$ to (4.2)-(4.3). However, for our subsequent discussion, we shall derive integral representations to u and $\tilde{u}$. To that end, we first recall the following celebrated gauge transformation (see, e.g. [24]). Let $L_\sigma u = \mathrm{div}(\sigma\nabla u)$. One has

$$\sigma^{-1/2}\circ L_\sigma \circ \sigma^{-1/2} = \Delta - \gamma, \quad \gamma = \sigma^{-1/2}\Delta\sigma^{1/2}. \tag{4.4}$$

Setting $v = \sigma_0^{1/2} u$ and using (4.4), (4.1) and (4.3) are transformed into

$$\begin{cases} (\Delta + \omega^2 q)v = 0 \quad \text{in } \ \mathbb{R}^n\backslash(\bar{U}_\rho \cup \bar{G}), \\ v|_{\partial U_\rho \cup \partial G} = 0, \end{cases} \tag{4.5}$$

where

$$q = \sigma_0^{-1}\eta_0 - \omega^{-2}\gamma_0, \quad \gamma_0 = \sigma_0^{-1/2}\Delta\sigma_0^{1/2}. \tag{4.6}$$

Moreover, $v^s := v - e^{ix\cdot\xi}$ satisfies the radiating condition (4.3). Obviously, $q \in C^1(\mathbb{R}^n)$ and $1 - q$ is compactly supported in $\mathbf{B}$. By introducing $\tilde{v} = \sigma_0^{1/2}\tilde{u}$ and $\tilde{v}^s = \tilde{v} - e^{ix\cdot\xi}$, we would have a similar system for $\tilde{v}$ and $\tilde{v}_s$, which should be clear in the context.

We shall make essential use of integral representations to v and $\tilde{v}$ for our arguments. To that end, we briefly introduce some potential operators and fix some notations for our subsequent discussion. Let Ω be a bounded C^2-domain in $\mathbb{R}^n$. We write Ω^- as Ω and Ω^+ as its complementary, and $\Pi = \partial\Omega^+ = \partial\Omega^-$. The one-sided trace operators for Ω^+ and Ω^- are denoted by γ^+ and γ^-, respectively. The normal derivative of a function $u \in H^1(\Omega^\pm)$ is understood in the usual way as $\mathcal{B}_\nu^\pm = \sum_{j=1}^n \nu_j\gamma^\pm(\partial_j u)$, where the unit normal ν points out of Ω^- and into Ω^+. We shall drop the $+$ or $-$ superscript if the two-sided traces coincide. The adjoint operators γ^* and $\mathcal{B}_\nu^*$ are respectively defined by

$$(\gamma^*\psi, \phi) = (\psi, \gamma\phi)_\Pi \text{ for } \ \phi \in \mathscr{E}(\mathbb{R}^n) \text{ and } \psi \in H^{\epsilon-1}(\Pi),\ 0 < \epsilon \le 2, \tag{4.7}$$

and

$$(\mathcal{B}_\nu^*\psi, \phi) = (\psi, \mathcal{B}_\nu\phi)_\Pi \quad \text{for } \phi \in \mathscr{E}(\mathbb{R}^n) \text{ and } \psi \in L^1(\Pi). \tag{4.8}$$

Let

$$\Phi(x,y)=\frac{e^{i\omega|x-y|}}{4\pi|x-y|}\ \text{ for } n=3;\quad \frac{i}{4}H_0^{(1)}(\omega|x-y|)\ \text{ for } n=2, \tag{4.9}$$

with $H_0^{(1)}(t)$ the zeroth order Hankel function of the first kind be the fundamental solutions to $-\Delta-\omega^2$, respectively, in $\mathbb{R}^3$ and $\mathbb{R}^2$. Now, we define

$$\mathcal{G}u(x)=\int_{\mathbb{R}^n}\Phi(x,y)u(y)\,dy\ \text{ for } x\in\mathbb{R}^n, \tag{4.10}$$

and the *single-layer potential* SL and the *double-layer potential* by

$$SL=\mathcal{G}\gamma*\quad\text{ and }\quad DL=\mathcal{G}\mathcal{B}_\nu^*, \tag{4.11}$$

whose integral representations are given by

$$\begin{aligned} SL\psi(x)&=\int_\Pi\Phi(x,y)\psi(y)\ dy,\\ DL\psi(x)&=\int_\Pi\frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi(y)\ dy. \end{aligned} \tag{4.12}$$

In the sequel, for notational convenience, we shall write for a set $\mathcal{W}\subset\mathbb{R}^n$,

$$SL_{\Pi,\mathcal{W}}\psi(x):=\int_\Pi\Phi(x,y)\psi(y)\ dy\quad\text{for } y\in\mathcal{W},$$

and similarly for $DL_{\Pi,\mathcal{W}}$. Let

$$S_\Pi:=SL_{\Pi,\Pi},\quad D_\Pi:=DL_{\Pi,\Pi}$$

which are understood in the sense of improper integrals. We shall also need the *volume potentials*

$$\mathcal{K}\psi(x):=\int_{\mathbf{B}\backslash(\bar{U}_\rho\cup\bar{G})}\Phi(x,y)[1-q(y)]\psi(y)\ dy, \tag{4.13}$$

$$\widetilde{\mathcal{K}}\phi(x):=\int_{\mathbf{B}\backslash\bar{G}}\Phi(x,y)[1-q(y)]\phi(y)\ dy, \tag{4.14}$$

and

$$\mathcal{J}\zeta(x):=\int_{U_\rho}\Phi(x,y)[q(y)-1]\zeta(y)\ dy \tag{4.15}$$

We refer to [4, 19] for mapping properties of the potential operators introduced here. Finally, we let

$$\Omega=\mathbf{B}\backslash(\bar{U}_\rho\cup\bar{G}),\quad\tilde{\Omega}=\mathbf{B}\backslash\bar{G},$$

and

$$\partial G := \Sigma; \quad \Gamma_{k,\rho} := \partial U_{k,\rho}, \quad \Gamma_\rho := \bigcup_{k=1}^{l} \Gamma_{k,\rho} = \partial U_\rho;$$

$$\Gamma_k := \partial U_k, \ \Gamma := \bigcup_{k=1}^{l} \Gamma_k = \partial U.$$

Lemma 4.1. *Let $v^i(x) := e^{ix\cdot\xi}$. The solution $v \in C^2(\mathbb{R}^n \backslash (\bar{U}_\rho \cup \bar{G})) \cap C(\mathbb{R}^n \backslash (U_\rho \cup G))$ to (4.5) is given by*

$$\begin{aligned} v(x) =& v^i(x) - \omega^2 \int_{\mathbb{R}^n \backslash (\bar{U}_\rho \cup \bar{G})} \Phi(x,y)[1-q(y)]v(y)\ dy \\ &+ \int_\Sigma \left[\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\Phi(x,y)\right] \psi_1(y)\ dy \\ &+ \int_{\Gamma_\rho} \left[\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\kappa\Phi(x,y)\right] \psi_2(y)\ dy, \end{aligned} \tag{4.16}$$

where $x \in \mathbb{R}^n \backslash (U_\rho \cup G)$ and $\kappa = [e(\rho)]^{-1}$. Here, $v|_{\bar{\Omega}} \in C(\bar{\Omega})$, $\psi_1 \in C(\Sigma)$ and $\psi_2 \in C(\Gamma_\rho)$ are uniquely determined by the following system of integral equations,

$$\begin{aligned} & v + \omega^2 \mathcal{K} v - (DL_{\Sigma,\bar{\Omega}} - iSL_{\Sigma,\bar{\Omega}})\psi_1 && (4.17)\\ & \qquad - (DL_{\Gamma_\rho,\bar{\Omega}} - i\kappa SL_{\Gamma_\rho,\bar{\Omega}})\psi_2 = p \quad in\ \bar{\Omega}, \\ & \frac{1}{2}\psi_1 - \omega^2\gamma\mathcal{K}v + (D_\Sigma - iS_\Sigma)\psi_1 && (4.18)\\ & \qquad + (DL_{\Gamma_\rho,\Sigma} - i\kappa SL_{\Gamma_\rho,\Sigma})\psi_2 = q_1 \quad on\ \Sigma, \\ & \frac{1}{2}\psi_2 - \omega^2\gamma\mathcal{K}v + (DL_{\Sigma,\Gamma_\rho} - iSL_{\Sigma,\Gamma_\rho})\psi_1 && (4.19)\\ & \qquad + (D_{\Gamma_\rho} - i\kappa S_{\Gamma_\rho})\psi_2 = q_2 \quad on\ \Gamma_\rho, \end{aligned}$$

where $p(x) = v^i(x)$ for $x \in \bar{\Omega}$, $q_1(x) = -p(x)$ for $x \in \Sigma$ and, $q_2(x) = -p(x)$ for $x \in \Gamma_\rho$. The system (4.17)–(4.19) is uniquely solvable in $C(\bar{\Omega}) \times C(\Sigma) \times C(\Gamma_\rho)$ for every $p \in C(\bar{\Omega})$, $p_1 \in C(\Sigma)$ and $p_2 \in C(\Gamma_\rho)$. It is also uniquely solvable in $L^2(\Omega) \times C(\Sigma) \times C(\Gamma_\rho)$ for every $p \in L^2(\Omega)$ and, $p_1 \in C(\Sigma)$ and $p_2 \in C(\Gamma_\rho)$.

We remark that a similar scattering problem to (4.5) is considered in [18], where the combination of volume potential and boundary layer potentials are implemented to represent the scattered wave field. However, the main concern of our study is the scattering behaviors due to the asymptotically small obstacles U_ρ. The novelty of the integral representation (4.16) is the introduction of the asymptotically coupling

parameter κ for layer potentials on the boundaries of the small obstacle components, which shall be crucial for our subsequent scattering estimates. The proof of Lemma 4.1 follows from a similar manner to that in [18], which for completeness we briefly include in the following.

Proof of Lemma 4.1. Using the mapping properties of potential operators (cf. [4]), along with the fact that $(-\Delta_y - \omega^2)\Phi(x,y) = \delta_x$, it is straightforward to show that for v given in (4.16), $v \in C^2(\mathbb{R}^n\backslash(\bar{U}_\rho \cup \bar{G})) \cap C(\mathbb{R}^n\backslash(U_\rho \cup G))$ satisfies $(\Delta + \omega^2 q)v = 0$. Whereas by the jump properties (cf. [4]), (4.18) implies $v|_\Sigma = 0$ and (4.19) implies $v|_{\Gamma_\rho} = 0$. The radiation condition for v^s is a direct consequence of the integral kernel $\Phi(x,y)$. So, we only need to show the well-posedness of the system (4.17)–(4.19), which could be written as

$$(\mathbf{A} + \mathbf{K})\mathbf{u} = \mathbf{p} \tag{4.20}$$

where

$$\mathbf{A} = \begin{bmatrix} I & -DL_{\Sigma,\bar{\Omega}} & -DL_{\Gamma_\rho,\bar{\Omega}} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}, \ \mathbf{u} = \begin{pmatrix} v \\ \psi_1 \\ \psi_2 \end{pmatrix}, \ \mathbf{p} = \begin{pmatrix} p \\ 2q_1 \\ 2q_2 \end{pmatrix} \tag{4.21}$$

and

$$\mathbf{K} = \begin{bmatrix} \omega^2\mathcal{K} & iSL_{\Sigma,\bar{\Omega}} & i\kappa SL_{\Gamma_\rho,\bar{\Omega}} \\ -2\omega^2\gamma\mathcal{K} & 2(D_\Sigma - iS_\Sigma) & 2(DL_{\Gamma_\rho,\Sigma} - i\kappa SL_{\Gamma_\rho,\Sigma}) \\ -2\omega^2\gamma\mathcal{K} & 2(DL_{\Sigma,\Gamma_\rho - iSL_{\Sigma,\Gamma_\rho}}) & 2(D_{\Gamma_\rho} - i\kappa S_{\Gamma_\rho}) \end{bmatrix}. \tag{4.22}$$

It is verified directly that

$$\mathbf{A}^{-1} = \begin{bmatrix} I & DL_{\Sigma,\bar{\Omega}} & DL_{\Gamma_\rho,\bar{\Omega}} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix},$$

and $\mathbf{K}$ is compact in both $L^2(\Omega) \times C(\Sigma) \times C(\Gamma_\rho)$ and $C(\bar{\Omega}) \times C(\Sigma) \times C(\Gamma_\rho)$. So, $\mathbf{A}+\mathbf{K}$ is an Fredholm operator of index 0. We only need to prove the uniqueness of the system (4.20), and it suffices to show this in $L^2(\Omega) \times C(\Sigma) \times C(\Gamma_\rho)$. Set $\mathbf{p} = 0$. By the uniqueness of solution to the scattering system (4.5), we see $v = 0$ in Ω. Let

$$\begin{aligned} \mathrm{w}(x) = & \int_\Sigma \left[\frac{\partial\Phi(x,y)}{\partial\nu(y)} - i\Phi(x,y)\right]\psi_1(y)\,dy \\ & + \int_{\Gamma_\rho}\left[\frac{\partial\Phi(x,y)}{\partial\nu(y)} - i\kappa\Phi(x,y)\right]\psi_2(y)\,dy, \quad x \in \mathbb{R}^n\backslash(\Sigma \cup \Gamma_\rho). \end{aligned}$$

Then, by (4.17), we see $\mathrm{w}|_\Omega = v|_\Omega = 0$. By the jump properties of layer potential operators, we have

$$-\gamma^-\mathrm{w} = \psi_1, \ -\gamma^-(\frac{\partial \mathrm{w}}{\partial \nu}) = i\psi_1 \quad \text{on } \Sigma;$$
$$-\gamma^-\mathrm{w} = \psi_2, \ -\gamma^-(\frac{\partial \mathrm{w}}{\partial \nu}) = i\kappa\psi_2 \quad \text{on } \Gamma_\rho;$$

By Green's formula,

$$i\int_\Sigma |\psi_1|^2 \ ds = \int_\Sigma (\gamma^-\mathrm{w})\gamma^-(\frac{\partial \mathrm{w}}{\partial \nu}) \ ds = \int_G |\nabla \mathrm{w}|^2 - \omega^2|\mathrm{w}|^2 \ dx,$$

which implies $\psi_1 = 0$. Similarly, one can show that $\psi_2 = 0$. The proof is completed.

□

Remark 4.2. Similar to Lemma 4.1, we know the solution $\tilde{v} \in C^2(\mathbb{R}^n\backslash\bar{G})\cap C(\mathbb{R}^n\backslash G)$ to the scattering problem without the small obstacle U_ρ is given by

$$\begin{aligned}\tilde{v}(x) =& v^i(x) - \omega^2\int_{\mathbb{R}^n\backslash\bar{G}} \Phi(x,y)[1-q(y)]\tilde{v}(y) \ dy \\ &+ \int_\Sigma \left[\frac{\partial\Phi(x,y)}{\partial\nu(y)} - i\Phi(x,y)\right]\tilde{\psi}(y) \ dy \quad x \in \mathbb{R}^n\backslash G,\end{aligned} \tag{4.23}$$

with $\tilde{v}|_{\bar{\tilde{\Omega}}} \in C(\bar{\tilde{\Omega}})$ and $\tilde{\psi} \in C(\Sigma)$ uniquely determined by the following system of integral equations,

$$\tilde{v} + \omega^2\tilde{\mathcal{K}}\tilde{v} - (DL_{\Sigma,\bar{\tilde{\Omega}}} - iSL_{\Sigma,\bar{\tilde{\Omega}}})\tilde{\psi} = \tilde{p} \quad \text{in } \bar{\tilde{\Omega}}, \tag{4.24}$$
$$\frac{1}{2}\tilde{\psi} - \omega^2\gamma\tilde{\mathcal{K}}\tilde{v} + (D_\Sigma - iS_\Sigma)\tilde{\psi} = \tilde{q} \quad \text{on } \Sigma, \tag{4.25}$$

where $\tilde{p}(x) = v^i(x)$ for $x \in \bar{\Omega}$, $\tilde{q}(x) = -\tilde{p}(x)$ for $x \in \Sigma$. For the subsequent use, we let

$$\begin{aligned}\tilde{\mathbf{L}} &= \left[\begin{array}{cc} I+\omega^2\tilde{\mathcal{K}} & -DL_{\Sigma,\bar{\tilde{\Omega}}} + iSL_{\Sigma,\bar{\tilde{\Omega}}} \\ -\omega^2\gamma\tilde{\mathcal{K}} & \frac{1}{2}I + D_\Sigma - iS_\Sigma \end{array}\right], \\ \tilde{\mathbf{v}} &= \left[\begin{array}{c}\tilde{v}\\ \tilde{\psi}\end{array}\right], \quad \tilde{\mathbf{x}} = \left[\begin{array}{c}\tilde{p}\\ \tilde{q}\end{array}\right].\end{aligned} \tag{4.26}$$

Clearly, we have

$$\tilde{\mathbf{v}} = \tilde{\mathbf{L}}^{-1}\tilde{\mathbf{x}}, \tag{4.27}$$

both in $L^2(\tilde{\Omega}) \times C(\Sigma)$ and $C(\bar{\tilde{\Omega}}) \times C(\Sigma)$.

We next derive the key lemma in proving Theorem 3.2. In the following, the analysis is based on the space dimension being 3. Later, we shall indicate the necessary modifications for the two dimensional case. Henceforth, for an operator $\Lambda : X \mapsto Y$ with X, Y being Banach spaces, we shall denote by $\|\Lambda\|_{\mathcal{L}(X,Y)}$ the corresponding operator norm.

Lemma 4.3. *For $\rho \to 0^+$, we have*

$$\|v - \tilde{v}\|_{L^2(\Omega)} = \|v - \tilde{v}\|_{L^2(\tilde{\Omega}\backslash U_\rho)} = \mathcal{O}(\rho) \tag{4.28}$$

and

$$\|\psi_1 - \tilde{\psi}\|_{C(\Sigma)} = \mathcal{O}(\rho) \quad \textit{and} \quad \|\psi_2\|_{C(\Gamma_\rho)} = \mathcal{O}(1). \tag{4.29}$$

Proof. The proof shall be proceeded in 6 steps.

Step I. We introduce $w = w_1\chi_{\tilde{\Omega}\backslash U_\rho} + w_2\chi_{U_\rho} \in L^2(\tilde{\Omega})$ such that

$$w_1 = v \quad \text{in} \quad \Omega = \tilde{\Omega}\backslash U_\rho, \tag{4.30}$$

and

$$\begin{aligned} w_2(x) + \omega^2 \int_{\tilde{\Omega}\backslash U_\rho} & \Phi(x,y)[1-q(y)]w_1(y)\,dy \\ & -(DL_{\Sigma,U_\rho} - iSL_{\Sigma,U_\rho})\psi_1(x) \\ & -(DL_{\Gamma_\rho,U_\rho} - i\kappa SL_{\Gamma_\rho,U_\rho})\psi_2(x) = p(x) \quad x \in U_\rho. \end{aligned} \tag{4.31}$$

By (4.31), (4.30) and (4.17), one can see that $w \in L^2(\tilde{\Omega})$ and $w_2 \in L^2(U_\rho)$ satisfy the following system of operator equations

$$\begin{aligned} w + \omega^2\widetilde{\mathcal{K}}w + \omega^2\mathcal{J}w_2 - & (DL_{\Sigma,\tilde{\Omega}} - iSL_{\Sigma,\tilde{\Omega}})\psi_1 \\ & -(DL_{\Gamma_\rho,\tilde{\Omega}} - i\kappa SL_{\Gamma_\rho,\tilde{\Omega}})\psi_2 = p \quad \text{in} \quad \tilde{\Omega}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} w_2 + \omega^2\mathcal{J}w_2 + \omega^2\widetilde{\mathcal{K}}w - & (DL_{\Sigma,U_\rho} - iSL_{\Sigma,U_\rho})\psi_1 \\ & -(DL_{\Gamma_\rho,U_\rho} - i\kappa SL_{\Gamma_\rho,U_\rho})\psi_2 = p \quad \text{in} \quad U_\rho. \end{aligned} \tag{4.33}$$

Step II. Let $\Theta(x) = x/\rho : \; U_\rho \cup \Gamma_\rho \to U \cup \Gamma$. In the sequel, for $x \in \Gamma$, we define

$$(S_\Gamma^0\phi)(x) = \int_\Gamma \Phi_0(x,y)\phi(y)\;dy, \quad (D_\Gamma^0\phi)(x) = \int_\Gamma \frac{\Phi_0(x,y)}{\partial\nu(y)}\phi(y)\;dy,$$

where $\Phi_0(x,y) = 1/(4\pi|x-y|)$ is the fundamental solution to $-\Delta$. For $\phi \in C(\Gamma_\rho)$, by using change of variables in integration, one has

$$\int_{\Gamma_\rho} \frac{e^{i\omega|x-y|}}{|x-y|}\phi(y)\;dy = \rho\int_\Gamma \frac{e^{i\omega\rho|x'-y'|}}{|x'-y'|}\phi(\rho y')\;dy',$$

$$\int_{\Gamma_\rho} \partial\left(\frac{e^{i\omega|x-y|}}{|x-y|}\right)/\partial\nu(y)\phi(y)\;dy = \int_\Gamma \partial\left(\frac{e^{i\omega\rho|x'-y'|}}{|x'-y'|}\right)/\partial\nu(y')\phi(\rho y')\;dy',$$

where $x' = x/\rho, y' = y/\rho \in \Gamma$. Using these along with power series expansion of $\exp\{i\omega\rho|x'-y'|\}$, one can verify directly that for $\phi \in C(\Gamma_\rho)$

$$\|(\Theta^{-1})^* \circ S_{\Gamma_\rho} \circ (\Theta^{-1})^*\phi - S_\Gamma^0 \circ (\Theta^{-1})^*\phi\|_{C(\Gamma)} \lesssim \rho\|\phi\|_{C(\Gamma_\rho)}, \tag{4.34}$$

where in the above inequality we have identified $\phi(x) \in C(\Gamma_\rho)$ as $\phi(x) = \phi(\rho x') = (\Theta^{-1})^*\phi \in C(\Gamma)$, and the fact that $\|\phi\|_{C(\Gamma_\rho)} = \|(\Theta^{-1})^*\phi\|_{C(\Gamma)}$. Similarly, we have

$$\|(\Theta^{-1})^* \circ D_{\Gamma_\rho} \circ (\Theta^{-1})^*\phi - D_\Gamma^0 \circ (\Theta^{-1})^*\phi\|_{C(\Gamma)} \lesssim \rho^2\|\phi\|_{C(\Gamma_\rho)}. \tag{4.35}$$

Next, by (4.19), together with the use of (4.34) and (4.35), we have

$$\begin{aligned}(\Theta^{-1})^*\psi_2 =&[\frac{1}{2}I + D_\Gamma^0 - iS_\Gamma^0 + \mathcal{O}(\rho)]^{-1}(\Theta^{-1})^*\Bigg[q_2 + \omega^2\gamma\mathcal{K}v\\ &- (DL_{\Sigma,\Gamma_\rho} - iSL_{\Sigma,\Gamma_\rho})\psi_1\Bigg].\end{aligned} \tag{4.36}$$

Here, we made use of the invertibility of $(\frac{1}{2}I + D_\Gamma^0 - iS_\Gamma^0) : C(\Gamma) \mapsto C(\Gamma)$ (cf. [5]).

Step III. Using (4.36), we consider

$$\begin{aligned}&(DL_{\Gamma_\rho,\Sigma} - i\kappa SL_{\Gamma_\rho,\Sigma})\psi_2\\ =&(DL_{\Gamma_\rho,\Sigma} - i\kappa SL_{\Gamma_\rho,\Sigma})\Theta^*\Bigg\{[\frac{1}{2}I + D_\Gamma^0 - iS_\Gamma^0 + \mathcal{O}(\rho)]^{-1}\\ &(\Theta^{-1})^*\Bigg[q_2 + \omega^2\gamma\mathcal{K}v - (DL_{\Sigma,\Gamma_\rho} - iSL_{\Sigma,\Gamma_\rho})\psi_1\Bigg]\Bigg\}\end{aligned} \tag{4.37}$$

We first note that

$$\mathrm{dist}_{\mathbb{R}^n}(\Gamma, \Sigma) < \mathrm{dist}_{\mathbb{R}^n}(\Gamma_\rho, \Sigma) < \mathrm{diam}_{\mathbb{R}^n}(\mathbf{B}),$$

and hence by direct verification

$$\|DL_{\Gamma_\rho,\Sigma} - i\kappa SL_{\Gamma_\rho,\Sigma}\|_{\mathcal{L}(C(\Gamma_\rho),C(\Sigma))} = \mathcal{O}(\rho) \tag{4.38}$$

and

$$\|DL_{\Sigma,\Gamma_\rho} - iSL_{\Sigma,\Gamma_\rho}\|_{\mathcal{L}(C(\Sigma),C(\Gamma_\rho))} = \mathcal{O}(1)). \tag{4.39}$$

By letting $\hat{v} = v\chi_\Omega + 0\chi_{U_\rho} \in L^2(\tilde{\Omega})$, and using the fact that $\widetilde{\mathcal{K}}$ maps $L^2(\tilde{\Omega})$ continuously into $H^2(\tilde{\Omega})$ (cf. [4]), we also see that

$$\begin{aligned}\|\mathcal{K}v\|_{C(\Omega)} \le \|\widetilde{\mathcal{K}}\hat{v}\|_{C(\tilde{\Omega})} \lesssim \|\widetilde{\mathcal{K}}\hat{v}\|_{H^2(\tilde{\Omega})}\\ \lesssim \|\hat{v}\|_{L^2(\tilde{\Omega})} = \|v\|_{L^2(\Omega)} \le \|w\|_{L^2(\tilde{\Omega})}.\end{aligned} \tag{4.40}$$

By (4.37)–(4.40), we have

$$(DL_{\Gamma_\rho,\Sigma} - i\kappa SL_{\Gamma_\rho,\Sigma})\psi_2 = \mathcal{A}_1(v) + \mathcal{A}_2(\psi_1) + \mathcal{A}_3(q_2), \tag{4.41}$$

where

$$\begin{aligned}\mathcal{A}_1(v) =&(DL_{\Gamma_\rho,\Sigma}-i\kappa SL_{\Gamma_\rho,\Sigma})\Theta^*\bigg([\frac{1}{2}I+D_\Gamma^0-iS_\Gamma^0\\ &+\mathcal{O}(\rho)]^{-1}(\Theta^{-1})^*(\omega^2\gamma\mathcal{K}v)\bigg)\end{aligned}\tag{4.42}$$

satisfying

$$\|\mathcal{A}_1(v)\|_{C(\Sigma)}\lesssim\rho\|v\|_{L^2(\Omega)}\lesssim\rho\|w\|_{L^2(\tilde{\Omega})};\tag{4.43}$$

and

$$\begin{aligned}\mathcal{A}_2(\psi_1) =&(DL_{\Gamma_\rho,\Sigma}-i\kappa SL_{\Gamma_\rho,\Sigma})\Theta^*\bigg([\frac{1}{2}I+D_\Gamma^0-iS_\Gamma^0+\mathcal{O}(\rho)]^{-1}\\ &(\Theta^{-1})^*(-DL_{\Sigma,\Gamma_\rho}+iSL_{\Sigma,\Gamma_\rho})\psi_1\bigg)\end{aligned}\tag{4.44}$$

satisfying

$$\|\mathcal{A}_2(\psi_1)\|\lesssim\rho\|\psi_1\|_{C(\Sigma)};\tag{4.45}$$

and

$$\begin{aligned}\mathcal{A}_3(q_2) =&(DL_{\Gamma_\rho,\Sigma}-i\kappa SL_{\Gamma_\rho,\Sigma})\Theta^*\\ &([\frac{1}{2}I+D_\Gamma^0-iS_\Gamma^0+\mathcal{O}(\rho)]^{-1}(\Theta^{-1})^*q_2)\end{aligned}\tag{4.46}$$

satisfying

$$\|\mathcal{A}_3(q_2)\|_{C(\Sigma)}\lesssim\rho\|q_2\|_{C(\Gamma_\rho)}.\tag{4.47}$$

Hence, we see from (4.47) that

$$\|\mathcal{A}_3(q_2)\|_{C(\Sigma)}\lesssim\rho.\tag{4.48}$$

Plugging (4.36) into (4.18), and using (4.37)–(4.48), we have

$$[\frac{1}{2}I+D_\Sigma-iS_\Sigma+\mathcal{O}(\rho)]\psi_1-\omega^2\gamma\mathcal{K}v+\mathcal{A}_1(v)=\tilde{q}+\mathcal{O}(\rho)\quad\text{on}\ \ \Sigma.\tag{4.49}$$

Step IV. In the sequel, we shall denote by DL^0 and SL^0 the double- and single-layer potentials with the integral kernel $\Phi(x,y)$ replaced by $\Phi_0(x,y)$. Also, we let

$$\mathcal{J}_0\zeta(x)=\int_U\Phi_0(x,y)[1-\hat{q}(y)]\zeta(y)\,dy,$$

where $\hat{q}=(\Theta^{-1})^*q\in L^\infty(U)$. We know that $I+\mathcal{J}_0$ is bounded invertible from $L^2(U)$ to itself and (cf. [4])

$$\|(I+\omega^2\mathcal{J}_0)^{-1}\|_{\mathcal{L}(L^2(U),L^2(U))}\le c(U,\|\hat{q}\|_{L^\infty(U)},\omega).\tag{4.50}$$

Clearly, the bound in (4.50) is independent of ρ. Now, using change of variables in integration and by a similar argument to that in Step II, we have from (4.33)

$$\begin{aligned}(\Theta^{-1})^* w_2 = &[I + \omega^2 \mathcal{J}_0 + \mathcal{O}(\rho)]^{-1} (\Theta^{-1})^* \Bigg[- \omega^2 \widetilde{\mathcal{K}} w \\ &+ (DL_{\Sigma,U_\rho} - iSL_{\Sigma,U_\rho})\psi_1 + (DL_{\Gamma_\rho,U_\rho} - i\kappa SL_{\Gamma_\rho,U_\rho})\psi_2 + p \Bigg] \\ =&\mathcal{M}_1(w) + \mathcal{M}_2(\psi_1) + \mathcal{M}_3(\psi_2) + \mathcal{M}_4(p).\end{aligned} \tag{4.51}$$

We next assess the $L^2(U)$-norms of $\mathcal{M}_1(w)$, $\mathcal{M}_2(\psi_1)$, $\mathcal{M}_3(\psi_2)$ and $\mathcal{M}_4(p)$, respectively. In (4.51),

$$\mathcal{M}_1(w) = [I + \omega^2 \mathcal{J}_0 + \mathcal{O}(\rho)]^{-1} (\Theta^{-1})^* [-\omega^2 \widetilde{\mathcal{K}} w],$$

which satisfies

$$\|\mathcal{M}_1(w)\|_{L^2(U)} \lesssim \|\widetilde{\mathcal{K}} w\|_{C(\tilde{\Omega})} \lesssim \|w\|_{L^2(\tilde{\Omega})}. \tag{4.52}$$

In (4.51).

$$\mathcal{M}_2(\psi_1) = [I + \omega^2 \mathcal{J}_0 + \mathcal{O}(\rho)]^{-1} (\Theta^{-1})^* [(DL_{\Sigma,U_\rho} - iSL_{\Sigma,U_\rho})\psi_1],$$

Noting

$$\mathrm{dist}_{\mathbb{R}^n}(\Sigma, \Gamma) < \mathrm{dist}_{\mathbb{R}^n}(\Sigma, \bar{U}_\rho) < \mathrm{diam}_{\mathbb{R}^n}(\mathbf{B}),$$

one can easily show that

$$\|(DL_{\Sigma,U_\rho} - iSL_{\Sigma,U_\rho})\psi_1\|_{C(U_\rho)} \lesssim \|\psi_1\|_{C(\Sigma)},$$

and hence

$$\|\mathcal{M}_2(\psi_1)\|_{L^2(U)} \lesssim \|\psi_1\|_{C(\Sigma)}. \tag{4.53}$$

For $\mathcal{M}_3(\psi_2)$, we have

$$\mathcal{M}_3(\psi_2) = [I + \omega^2 \mathcal{J}_0 + \mathcal{O}(\rho)]^{-1} (\Theta^{-1})^* [(DL_{\Gamma_\rho,U_\rho} - i\kappa SL_{\Gamma_\rho,U_\rho})\psi_2].$$

It is verified directly that

$$\begin{aligned}(DL_{\Gamma_\rho,U_\rho}\psi_2)(x) &= \int_{\Gamma_\rho} \partial \left(\frac{e^{i\omega|x-y|}}{|x-y|} \right) / \partial\nu(y) \psi_2(y)\, dy \\ &= \int_{\Gamma} \partial \left(\frac{e^{i\omega\rho|x'-y'|}}{|x'-y'|} \right) / \partial\nu(y') \psi_2(\rho y')\, dy', \\ (SL_{\Gamma_\rho,U_\rho}\psi_2)(x) &= \int_{\Gamma_\rho} \frac{e^{i\omega|x-y|}}{|x-y|} \psi_2(y)\, dy = \rho \int_{\Gamma} \frac{e^{i\omega\rho|x'-y'|}}{|x'-y'|} \psi_2(\rho y')\, dy',\end{aligned}$$

where $x \in \Gamma_\rho, y \in U_\rho$ and $x' \in \Gamma, y' \in U$. Hence, we have

$$[DL_{\Gamma_\rho,U_\rho} - i\kappa SL_{\Gamma_\rho,U_\rho}]\psi_2 = [DL^0_{\Gamma,U} + iSL^0_{\Gamma,U} + \mathcal{O}(\rho)](\Theta^{-1})^*\psi_2,$$

and from which we further have

$$\|\mathcal{M}_3(\psi_2)\|_{L^2(U)} \lesssim \|\psi_2\|_{C(\Gamma_\rho)}. \tag{4.54}$$

Finally, by

$$\mathcal{M}_4(p) = [I + \omega^2 \mathcal{J}_0 + \mathcal{O}(\rho)]^{-1}(\Theta^{-1})^* p,$$

we see

$$\|\mathcal{M}_4(p)\|_{L^2(U)} = \mathcal{O}(1). \tag{4.55}$$

Now, by (4.51)–(4.55), we have

$$\begin{aligned}
\|w_2\|_{L^2(U_\rho)} &= \rho^{3/2}\|(\Theta^{-1})^* w_2\|_{L^2(U)} \le \rho^{3/2}\Big(\|\mathcal{M}_1(w)\|_{L^2(U)} \\
&\quad + \|\mathcal{M}_2(\psi_1)\|_{L^2(U)} + \|\mathcal{M}_3(\psi_2)\|_{L^2(U)} + \|\mathcal{M}_4(p)\|_{L^2(U)}\Big) \\
&\lesssim \rho^{3/2}\|w\|_{L^2(\tilde\Omega)} + \rho^{3/2}\|\psi_1\|_{C(\Sigma)} + \rho^{3/2}\|\psi_2\|_{C(\Gamma_\rho)} + \mathcal{O}(\rho^{3/2}).
\end{aligned} \tag{4.56}$$

Step V. We now consider the equation (4.32). Let $\hat w = w_2\chi_{U_\rho} + 0\chi_\Omega \in L^2(\tilde\Omega)$. Using (4.56), we have

$$\begin{aligned}
\|\mathcal{J} w_2\|_{H^2(\tilde\Omega)} &= \|-\widetilde{\mathcal{K}}\hat w\|_{H^2(\tilde\Omega)} \lesssim \|\hat w\|_{L^2(\tilde\Omega)} = \|w_2\|_{L^2(U_\rho)} \\
&\lesssim \rho^{3/2}\|w\|_{L^2(\tilde\Omega)} + \rho^{3/2}\|\psi_1\|_{C(\Sigma)} + \rho^{3/2}\|\psi_2\|_{C(\Gamma_\rho)} + \mathcal{O}(\rho^{3/2}).
\end{aligned} \tag{4.57}$$

By (4.36), (4.39) and (4.40), one can show

$$\begin{aligned}
\|\psi_2\|_{C(\Gamma_\rho)} &= \|(\Theta^{-1})^*\psi_2\|_{C(\Gamma)} \\
&\lesssim \|q_2\|_{C(\Gamma_\rho)} + \|w\|_{L^2(\tilde\Omega)} + \|\psi_1\|_{C(\Sigma)} \\
&\lesssim \|\tilde q\|_{C(\tilde\Omega)} + \|w\|_{L^2(\tilde\Omega)} + \|\psi_1\|_{C(\Sigma)} + \mathcal{O}(\rho),
\end{aligned} \tag{4.58}$$

which together with (4.56) and (4.57) implies

$$\|w_2\|_{L^2(U_\rho)} \lesssim \ \rho^{3/2}\|w\|_{L^2(\tilde\Omega)} + \rho^{3/2}\|\psi_1\|_{C(\Sigma)} + \mathcal{O}(\rho^{3/2}), \tag{4.59}$$

and

$$\|\mathcal{J} w_2\|_{H^2(\tilde\Omega)} \lesssim \ \rho^{3/2}\|w\|_{L^2(\tilde\Omega)} + \rho^{3/2}\|\psi_1\|_{C(\Sigma)} + \mathcal{O}(\rho^{3/2}). \tag{4.60}$$

Let $\vartheta \in \mathscr{D}(\mathbb{R}^3)$ be a cut-off function satisfying $\vartheta = 1$ on $\mathbf{B}$ and $\vartheta = 0$ on $\mathbf{D}$ with $\mathbf{D}$ a neighborhood of $\mathbf{B}$. For $\phi \in \mathscr{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&(DL_{\Gamma_\rho,\mathbb{R}^n}\psi_2, \vartheta\phi)_{L^2(\mathbb{R}^n)} = (\mathcal{G}\vartheta\mathcal{B}_\nu^*\psi_2, \vartheta\phi)_{L^2(\mathbb{R}^n)} \\
=&(\psi_2, \mathcal{B}_\nu(\vartheta\mathcal{G}^*\vartheta)\phi)_{L^2(\Gamma_\rho)} \le \|\psi_2\|_{L^2(\Gamma_\rho)}\|\mathcal{B}_\nu(\vartheta\mathcal{G}^*\vartheta)\phi\|_{L^2(\Gamma_\rho)} \\
\lesssim& \rho\|\psi_2\|_{C(\Gamma_\rho)}\|(\vartheta\mathcal{G}^*\vartheta)\phi\|_{H^2(\mathbf{D})} \\
\lesssim& \rho\|\psi_2\|_{C(\Gamma_\rho)}\|\vartheta\phi\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{4.61}$$

and from which one easily see that

$$\|DL_{\Gamma_\rho,\tilde\Omega}\psi_2\|_{L^2(\tilde\Omega)} \lesssim \rho\|\psi_2\|_{C(\Gamma_\rho)}. \tag{4.62}$$

Here, we have made use of the fact that $\mathcal{G}^*$ maps $L^2(\mathbf{D})$ continuously into $H^2(\mathbf{D})$. Similarly, we have

$$\begin{aligned}
&(SL_{\Gamma_\rho,\mathbb{R}^n}\psi_2,\vartheta\phi)_{L^2(\mathbb{R}^n)}=(\mathcal{G}\vartheta\gamma^*\psi_2,\vartheta\phi)_{L^2(\mathbb{R}^n)}\\
=&(\psi_2,\gamma(\vartheta\mathcal{G}^*\vartheta)\phi)_{L^2(\Gamma_\rho)}\leq\|\psi_2\|_{L^2(\Gamma_\rho)}\|\gamma(\vartheta\mathcal{G}^*\vartheta)\phi\|_{L^2(\Gamma_\rho)}\\
\lesssim&\,\rho^2\|\psi_2\|_{C(\Gamma_\rho)}\|(\vartheta\mathcal{G}^*\vartheta)\phi\|_{C(\Gamma_\rho)}\\
\lesssim&\,\rho^2\|\psi_2\|_{C(\Gamma_\rho)}\|(\vartheta\mathcal{G}^*\vartheta)\phi\|_{H^2(\mathbf{D})}\\
\lesssim&\,\rho^2\|\psi_2\|_{C(\Gamma_\rho)}\|\vartheta\phi\|_{L^2(\mathbb{R}^n)},
\end{aligned}\tag{4.63}$$

and from which one has

$$\|\kappa SL_{\Gamma_\rho,\tilde\Omega}\psi_2\|_{L^2(\tilde\Omega)}=\rho^{-1}\|SL_{\Gamma_\rho,\tilde\Omega}\psi_2\|_{L^2(\tilde\Omega)}\lesssim\rho\|\psi_2\|_{C(\Gamma_\rho)}.\tag{4.64}$$

By (4.62) and (4.64), we see that

$$\|(DL_{\Gamma_\rho,\tilde\Omega}-i\kappa SL_{\Gamma_\rho,\tilde\Omega})\psi_2\|_{L^2(\tilde\Omega)}\lesssim\rho\|\psi_2\|_{C(\Gamma_\rho)},\tag{4.65}$$

which together with (4.58) further gives

$$\begin{aligned}
&\|(DL_{\Gamma_\rho,\tilde\Omega}-i\kappa SL_{\Gamma_\rho,\tilde\Omega})\psi_2\|_{L^2(\tilde\Omega)}\\
\lesssim&\,\rho\|w\|_{L^2(\tilde\Omega)}+\rho\|\psi_1\|_{C(\Sigma)}+\mathcal{O}(\rho).
\end{aligned}\tag{4.66}$$

Obviously, (4.60) implies that

$$\begin{aligned}
&\|\mathcal{J}w_2\|_{L^2(\tilde\Omega)}\leq\|\mathcal{J}w_2\|_{H^2(\tilde\Omega)}\\
\lesssim&\,\rho^{3/2}\|w\|_{L^2(\tilde\Omega)}+\rho^{3/2}\|\psi_1\|_{C(\Sigma)}+\mathcal{O}(\rho^{3/2}).
\end{aligned}\tag{4.67}$$

Finally, (4.32), (4.66) and (4.67), together with the fact that $q=\tilde q$ in $\tilde\Omega$, yield that $(w,\psi_1)\in L^2(\tilde\Omega)\times C(\Sigma)$ satisfies

$$\begin{aligned}
&[I+\omega^2\widetilde{\mathcal{K}}+\mathcal{O}(\rho)]w-[DL_{\Sigma,\tilde\Omega}-iSL_{\Sigma,\tilde\Omega}+\mathcal{O}(\rho)]\psi_1\\
=&\tilde p+\mathcal{O}(\rho)\quad\text{in }\tilde\Omega.
\end{aligned}\tag{4.68}$$

Step VI. In the following, we further assess (4.49). We first note

$$\gamma\mathcal{K}v=\gamma\widetilde{\mathcal{K}}w+\gamma\mathcal{J}w_2\quad\text{on }\Sigma.\tag{4.69}$$

By (4.57) and (4.58), one can show

$$\begin{aligned}
&\|\gamma\mathcal{J}w_2\|_{C(\Sigma)}=\|-\gamma\widetilde{\mathcal{K}}\hat w\|_{C(\Sigma)}\lesssim\|\widetilde{\mathcal{K}}\hat w\|_{H^2(\tilde\Omega)}\\
\lesssim&\,\rho^{3/2}\|w\|_{L^2(\tilde\Omega)}+\rho^{3/2}\|\psi_1\|_{C(\Sigma)}+\mathcal{O}(\rho^{3/2}),
\end{aligned}\tag{4.70}$$

which together with (4.69), (4.49) and (4.43) implies

$$[\frac{1}{2}I+D_\Sigma-iS_\Sigma+\mathcal{O}(\rho)]\psi_1-[\omega^2\gamma\mathcal{K}+\mathcal{O}(\rho)]w=\tilde q+\mathcal{O}(\rho)\quad\text{on}\ \ \Sigma.\tag{4.71}$$

Now, by (4.68) and (4.71) we see that $\mathbf{v} := (w, \psi_1) \in L^2(\tilde{\Omega}) \times C(\Sigma)$ satisfies

$$[\tilde{\mathbf{L}} + \mathcal{O}(\rho)]\mathbf{v} = \tilde{\mathbf{x}} + \mathcal{O}(\rho), \tag{4.72}$$

which by Remark 4.2 gives

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathcal{O}(\rho). \tag{4.73}$$

That is,

$$\|w - \tilde{v}\|_{L^2(\tilde{\Omega})} = \mathcal{O}(\rho) \quad \text{and} \quad \|\psi_1 - \tilde{\psi}\|_{C(\Sigma)} = \mathcal{O}(\rho), \tag{4.74}$$

which together with (4.58) implies

$$\|\psi_2\|_{C(\Gamma_\rho)} = \mathcal{O}(1). \tag{4.75}$$

It is trivially pointed out that

$$\|v - \tilde{v}\|_{L^2(\Omega)} \le \|w - \tilde{v}\|_{L^2(\tilde{\Omega})} = \mathcal{O}(\rho). \tag{4.76}$$

The proof is completed.

□

Remark 4.4. In two dimensions, we shall have as $\rho \to 0^+$

$$\|v - \tilde{v}\|_{L^2(\Omega)} = \mathcal{O}((\ln \rho)^{-1}) \tag{4.77}$$

and

$$\|\psi_1 - \tilde{\psi}\|_{C(\Sigma)} = \mathcal{O}((\ln \rho)^{-1}) \quad \text{and} \quad \|\psi_2\|_{C(\Gamma_\rho)} = \mathcal{O}(1). \tag{4.78}$$

The proof is very similar to that for the three dimensional case in Lemma 4.3, and the only modification we shall need is that at certain point, we shall make use of the following asymptotic expansion for $\Phi(x, y)$,

$$\begin{aligned} \Phi(x,y) =& \frac{i}{4} H_0^{(1)}(\omega|x-y|) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} + \frac{i}{4} \\ & - \frac{1}{2\pi} \ln \frac{\omega}{2} - \frac{E}{2\pi} + \mathcal{O}\left(|x-y|^2 \ln \frac{1}{|x-y|}\right) \end{aligned}$$

as $|x - y| \to 0$. Here, E denotes the Euler's constant.

Proof of Theorem 3.2. By noting $u^s = v^s$ in $\mathbb{R}^n \backslash \bar{\mathbf{B}}$ and (4.16), one has by direct asymptotic expansion that

$$\begin{aligned} A(\theta, \xi; \mathcal{T}) = & -\omega^2 \tau_0 \int_\Omega e^{-i\omega\theta\cdot y} [1 - q(y)] v(y)\, dy \\ & + \tau_0 \int_\Sigma \left[\frac{\partial e^{-i\omega\theta\cdot y}}{\partial \nu(y)} - i e^{-i\omega\theta\cdot y} \right] \psi_1(y)\, dy \\ & + \tau_0 \int_{\Gamma_\rho} \left[\frac{\partial e^{-i\omega\theta\cdot y}}{\partial \nu(y)} - i\kappa e^{-i\omega\theta\cdot y} \right] \psi_2(y)\, dy, \end{aligned} \tag{4.79}$$

where $\tau_0 = e^{i\frac{\pi}{4}}/\sqrt{8\pi\omega}$ for $n = 2$; and $\tau_0 = \frac{1}{4\pi}$ for $n = 3$. Similarly, by $\tilde{u}^s = \sigma_0^{-1/2}\tilde{v}^s$ and (4.23), one also has

$$\begin{aligned}
&A(\theta, \xi; G \oplus \{\mathbf{B}\backslash\bar{G}; \sigma_0, \eta_0\}) \\
= &-\omega^2\tau_0 \int_{\tilde{\Omega}} e^{-i\omega\theta\cdot y}[1 - q(y)]\tilde{v}(y)\, dy \\
&+ \tau_0 \int_{\Sigma} \left[\frac{\partial e^{-i\omega\theta\cdot y}}{\partial\nu(y)} - ie^{-i\omega\theta\cdot y}\right] \tilde{\psi}(y)\, dy
\end{aligned} \tag{4.80}$$

Subtracting (4.80) from (4.79), we have

$$\begin{aligned}
&A(\theta, \xi; \mathcal{T}) - A(\theta, \xi; G \oplus \{\mathbf{B}\backslash\bar{G}; \sigma_0, \eta_0\}) \\
= &-\omega^2\tau_0 \int_{\Omega} e^{-i\omega\theta\cdot y}[1 - q(y)][v(y) - \tilde{v}(y)]\, dy \\
&+ \tau_0 \int_{\Sigma} \left[\frac{\partial e^{-i\omega\theta\cdot y}}{\partial\nu(y)} - ie^{-i\omega\theta\cdot y}\right] [\psi_1(y) - \tilde{\psi}(y)]\, dy \\
&+ \tau_0 \int_{\Gamma_\rho} \left[\frac{\partial e^{-i\omega\theta\cdot y}}{\partial\nu(y)} - i\kappa e^{-i\omega\theta\cdot y}\right] \psi_2(y)\, dy \\
&+ \omega^2\tau_0 \int_{U_\rho} e^{-i\omega\theta\cdot y}[1 - q(y)]\tilde{v}(y)\, dy.
\end{aligned} \tag{4.81}$$

Applying Lemma 4.3 and Remark 4.4 to estimating the terms in (4.81), along with straightforward calculations, we have (3.14). The proof is completed. $\square$

We are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $u \in H^1_{loc}(\mathbb{R}^n\backslash(\bar{P}\cup\bar{G}))$ be the scattering wave field corresponding to $\mathcal{E} = \bigoplus_{k=1}^l \{D_k\backslash\bar{P}_k; \sigma_{k,c}, \eta_{k,c}\} \oplus P \oplus G \oplus \{\mathbf{B}\backslash(\bar{D} \cup \bar{G}); \sigma_0, \eta_0\}$, and $\tilde{u} \in H^1_{loc}(\mathbb{R}^n\backslash\bar{G})$ be the scattering wave field corresponding to $G \oplus \{\mathbf{B}\backslash\bar{G}; \sigma_0, \eta_0\}$. Define $F : \mathbb{R}^n\backslash(\bar{U}_\rho \cup \bar{G}) \to \mathbb{R}^n\backslash(\bar{P} \cup \bar{G})$ by

$$F := \begin{cases} F_{k,\rho} & \text{on } \bar{D}_k\backslash U_\rho, \\ \text{Identity} & \text{otherwise.} \end{cases} \tag{4.82}$$

Obviously, F is bi-Lipschitz and orientation-preserving. By the transformation invariance of the Helmhlotz equation as we discussed in Section 3, one sees that $F^*u \in H^1_{loc}(\mathbb{R}^n\backslash(\bar{U}_\rho \cup \bar{G}))$ is the scattering wave field corresponding to $\mathcal{T} = U_\rho \oplus G \oplus \{\mathbf{B}\backslash(\bar{U}_\rho \cup \bar{G}); \sigma_0, \eta_0\}$. Since F is the identity outside $D = \cup_{k=1}^l D_k$, we know

$$A(\theta, \xi; \mathcal{E}) = A(\theta, \xi; \mathcal{T}). \tag{4.83}$$

According to our discussion in Lemma 4.1 and Remark 4.2, we know $\tilde{u} \in C^2(\mathbb{R}^n\backslash\bar{G})$ and $F^*u \in C^2(\mathbb{R}^n\backslash(\bar{U}_\rho \cup \bar{G})) \cap C(\mathbb{R}^n\backslash(U_\rho \cup G))$. Then, by Theorem 3.2

$$\begin{aligned}&\|A(\theta,\xi;\mathcal{T}) - A(\theta,\xi; G \oplus \{\mathbf{B}\backslash\bar{G};\sigma_0,\eta_0\})\|_{L^2(\mathbb{S}^{n-1})}\\ =&\mathcal{O}(e(\rho)) \text{ as } \rho \to 0^+,\end{aligned}$$

which together with (4.83) yields

$$\|A(\theta,\xi;\mathcal{E}) - A(\theta,\xi; G \oplus \{\mathbf{B};\sigma_0,\eta_0\})\|_{L^2(\mathbb{S}^{n-1})} = \mathcal{O}(e(\rho)).$$

The proof is completed. □

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